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Q-operation in strong semilatticess of monoids

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Abstract. The purpose of this paper is to introduce a new class of strong semilatticess of monoids, the so called **q** -partial monoids. We show that every strong semilatticess of monoids S with identity contains a unique maximal q-strong semilattices of monoids $\mathrm{Q}(S)$ such that $(\mathrm{Q}(S))_1 = S_1$. Finally we show this Q-operation commute with cartesian products.

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1. Introduction

One can easily observe that Clifford semigroups have been the object of extensive study from both category and semigroup theorists. A Clifford semigroup is usually defined as a regular semigroup with central idempotents (see [1, 2, 4, 7]). Whence many characterizations exist including the structure theorem that characterizes them as semilattices of groups, or equivalently as strong semilattices of groups. That is if $S = [Y, S_\alpha, \psi_{\alpha,\beta}]$ is a strong semilattice Y of groups S_α, then S is a Clifford semigroup with operation defined by

$$ab = (\psi_{\alpha,\alpha\beta} a)(\psi_{\beta,\alpha\beta} b). \tag{1}$$

for $a \in S_\alpha, b \in S_\beta$.

Conversely, a Clifford semigroup S is a strong semilattice $E(S)$ of groups S_f; $S = [E(S), S_f, \psi_{f,g}]$ where $E(S)$ is the semilattice ($f \leq g \Leftrightarrow fg = f$) of idempotents in S, S_f is the maximal subgroup of S with identity f, and $\psi_{f,g}$ is the homomorphism $S_f \to S_g$, $a \mapsto ag$ if $f \geq g$. Here we observe that S may be viewed as a category with objects all S_f , $f \in E(S)$ (are objects in the category of groups) and arrows, also called morphisms, given for two objects S_f and S_g as follows: $Hom(S_f, S_g) = \{\psi_{f,g}\}$ if $f \geq g$ and $Hom(S_f, S_g) = \varnothing$ otherwise. In the main section A q Strong semilattices of monoids is defined to be a strong semilattices of monoids $S = [E(S), S_e, \ \psi_{e,f}]$ such that S has an identity 1 and $\varphi_{1,e}$ is an epimorphism for all $e \in E(S)$. Every Strong semilatticess of monoids S with identity contains a unique maximal q-strong semilatticess of monoids Q(S)such that $(\mathrm{Q}(S))_1 = S_1$. This Q-operation is proved to commute with cartesian products.

2. Strong semilattices of monoids

A lower semilattice is a partially ordered set $< X, \leq >$ for which $a \wedge b$ (the greatest lower bound of a and b) exists for all $a, b \in X$. If the greatest lower bound of every non-empty subset of X exists, then $< X, \leq >$, or simply X is called a complete lower semilattice. A complete upper semilattice is defined dually. A (complete) lower and upper semilattice $< L, \leq, \wedge, \vee >= L$ is called a (complete) lattice.

A sublattice of L is a non-empty subset M of L such that $a, b \in M$ implies $a \wedge b$, $a \vee b \in M$. If L and K are lattices, a mapping $\phi : L \to K$ is a homomorphism of lattices if $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ and $\phi(x \vee y) = \phi(x) \vee \phi(y)$ for all $x, y \in L$.
If moreover, ϕ is one -to-one and onto then ϕ is called an isomorphism of lattices.

Proposition 2.1 ([3])**.** *If $< E, \leq, \wedge >$ is a lower semilattice, then $< E, \wedge >$ is a commutative semigroup of idempotents and for all $a, b \in E$, $a \leq b$ if and only if $a \wedge b = a$. Conversely, if $< E, \cdot >$ is a commutative semigroup of idempotents, then the relation $\leq$ defined by $a \leq b$ if and only if $a \cdot b = a$ turns E into a lower semilattice in which $a \wedge b = a \cdot b$ for all $a, b \in E$.*

In view of Proposition 2.1 a lower semilattice is precisely a commutative semigroup of idempotents (commutative band) and usually referred to a semilattice (cf, [2]).

A lattice $L =< L, \leq, \wedge, \vee >$ is called modular if for all $a, b, c \in L, a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$. Since in any lattice we always have

$$a \leq c \text{ implies } a \vee (b \wedge c) \leq (a \vee b) \wedge c,$$

it follows that a lattice L is modular if and only if $a \leq c$ implies $a \vee (b \wedge c) \geq (a \vee b) \wedge c$.

Proposition 2.2 ([3])**.** *Let $\phi : S \to Y$ be a homomorphism from a semigroup S onto a semilattice Y. For each $\alpha \in Y$ let $S_\alpha = \phi^{-1}(\alpha)$. Then*

(i) *S_α is a subsemigroup of S for all $\alpha \in Y$,*
(ii) *$S = \bigcup_{\alpha \in Y} S_\alpha$ and this union is disjoint,*
(iii) *$S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$.*

A semilattice of semigroups of type $\Im$ is a semigroup S which is a disjoint union of subsemigroups S_α, $\alpha \in Y$ of type $\Im$ indexed by a semilattice Y such that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$.

Definition 2.3 ([5])**.** Let Y be a semilattice and let $\{S_\alpha : \alpha \in Y\}$ be a family of monoids indexed by Y. For every $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ be a homomorphism such that

(a) $\phi_{\alpha,\alpha} : S_\alpha \to S_\alpha$ is the identical automorphism of S_α for all $\alpha \in Y$,
(b) $\phi_{\beta,\gamma}\, \phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$ for all $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$.

Then $S = \cup\{S_\alpha : \alpha \in Y\}$ is a strong semilattices of monoids.

Now we give examples of partial monoids.

Example 2.4 ([6])**.** Recall that a partial mapping from a set X into a set Y, written $\phi : X \rightarrowtail Y$, is a mapping ϕ from a (possibly empty) subset of X into Y, and therefore determined by a subset $B \subseteq X$ and a mapping $\phi : B \to Y$ with $dom\phi = B$. The set of all partial mappings from X into Y is denoted by $P(X, Y)$. This is actually a disjoint union of sets of ordinary mappings

$$P(X, Y) = \cup \{M(B, Y) : B \subseteq X\}$$

where $M(B, Y)$ is the set of all mappings from B into Y. Now let X be a set and let G be a monoid. Consider the set $P(X, G)$ of all partial mappings $\alpha : X \rightarrowtail G$. Thus, $P(X, G)$ is a disjoint union of groups $P(X, G) = \cup \{M(B, G) : B \subseteq X\}$, where $M(B, G)$ is the group of ordinary mappings from B into G whose binary operation is the multiplication of mappings $(\alpha_B \circ \gamma_B)x = (\alpha_B x)(\gamma_B x)$ for all $x \in B$. The identity of the monoid $M(B, G)$ is the (zero) mapping $e_B : B \to G, x \mapsto 1_G$. Define $E(P(X, G)) = \{e_B : B \subseteq X\}$. Then $E(P(X, G))$ is clearly a commutative semigroup of idempotents with the operation Δ defined as follows: $e_A \Delta e_B = e_{A\cap B}$ for all $A, B \subseteq X$. Equivalently, $E(P(X, G))$ is a semilattice with $e_A \leq e_B$ if and only if $A \subseteq B$. For any two idempotents e_A and e_B with $e_A \leq e_B$ define a mapping $\varphi_{B,A} : M(B, G) \to M(A, G)$ which sends each $\alpha_B : B \to G$ to its restriction $\alpha_A : A \to G$ on A. Clearly, $\varphi_{B,A}$ is a homomorphism of monoids, $\varphi_{B,B}$ is the identical automorphism of $M(B, G)$ and $e_C \geq e_B \geq e_A$ implies $\varphi_{B,A} \circ \varphi_{C,B} = \varphi_{C,A}$. It follows that $< P(X, G), \Delta >$ is a partial monoid with the binary operation defined for all $\alpha_A \in M(A, G)$, and $\gamma_B \in M(B, G)$ by

$$\alpha_A \Delta \gamma_B = (\varphi_{A,A\cap B}\alpha_A)(\varphi_{B,A\cap B}\gamma_B) = \alpha_{A\cap B}.\gamma_{A\cap B}$$

that is

$$\alpha_A \Delta \gamma_B : A \cap B \to G,\ x \mapsto \alpha_A(x).\gamma_B(x).$$

We denote $< P(X, G), \Delta >$ by $\overset{\Delta}{P}(X, G)$.

Example 2.5 ([6])**.** Suppose that $\{S_e, S_f, S_g, ...\}$ is a countable family of disjoint isomorphic monoids indexed, without loss of generality, by the set E of their identities. Let $S = \cup \{S_e : e \in E\}$. There exist two cases.

(a) E is countably infinite: In this case the set of all bijections $N \to E$ is countably infinite and an element in this set may be viewed as a sequence or a chain of $E-$elements. Let one of such countable sequences, say $< e_1, e_2, ..., e_i, ... >$. Thus E is turned into a countable chain $e_1 \leq e_2 \leq ...$, that is a commutative semigroup of idempotents (i.e. semilattice) with operation $e_i \cdot e_j = e_i$ if and only if $e_i \leq e_j$. By the assumption (for $e_i \leq e_j$) there exists a homomorphism

$$\varphi_{e_j,e_i} : S_{e_j} \to S_{e_i}$$

which is also an isomorphism. These isomorphisms can be chosen in such a way that ϕ_{e_i,e_i} is the identical automorphism of S_{e_i} and $\varphi_{e_j,e_k} \circ \varphi_{e_i,e_j} = \varphi_{e_i,e_k}$ for all e_i, e_j and e_k with $e_i \geq e_j \geq e_k$. Thus $S = [E(S), S_{e_i}, \varphi_{e_i,e_j}]$ is a strong semilattice of monoids There exist as many as countably infinite number of strong semilattice of monoids constructed in this way.

(b) E is finite : This is a special case of (a) above. If say, the number of elements in E is n, there exist exactly $n!$ different chains of E-elements. Thus by the above construction, $S = \cup \{S_{e_i} : i = 1, ..., n\}$ is strong semilattice of monoids in $n!$ ways.

3. q-strong semilattices of monoids

Throughout this section, S stands for strong semilattices of monoids with identity 1. Thus $S = [E(S); S_f, \varphi_{f,g}]$ with $E(S)$ having upper bound 1. We call the identity 1 of S *proper* if the maximal subgroup S_1 is not the trivial monoid , that is, if $\{1\}$ is a proper subset of S_1.Otherwise 1 is called *improper*. In the usual partial ordering of $E(S)$, we then have $1 \geq e$ for all $e \in E(S)$, and so we have a homomorphism of monoids $\varphi_{1,e} : S_1 \to S_e$ for every $e \in E(S)$.

We call S a **q**-strong semilattices of monoids if $\varphi_{1,e}$ is an epimorphism for every $e \in E(S)$, this is equivalent to say that $S_1 e = S_e$ for every $e \in E(S)$, that is, every $x \in S_e$ can be written as a product ye for some $y \in S_1$. Since $S_1 e \subset S_e$ always holds, S is a **q**-strong semilattices of monoids iff $S_e \subset S_1 e$ for all $e \in E(S)$.

If S is **q**-strong semilattices of monoids and $e \geq f$ in $E(S)$, then we have $\varphi_{e,f}(S_e) = S_e f = (S_1 e)f = S_1(ef) = S_1 f = S_f$. It follows that in a **q**-strong semilattices of monoids , every homomorphism $\varphi_{e,f}$ is an epimorphism. We observe also that in a **q**-strong semilattices of monoids S, $S_e \cdot S_f = S_{ef}$ $\forall e, f \in E(S)$. For, we have $S_e \cdot S_f \subset S_{ef}$ since S is a strong semilattice of its maximal monoids, and if $x \in S_{ef}$ then $x = yef$ for some $y \in S_1$ which gives $x = (ye)(1f) \in S_e \cdot S_f$.

Let T be a wide subset of S , T is called a q-strong semilattices of monoids of S if the restriction $\varphi_{1,e}$ on T_1 is epimorphism for every $e \in E(S)$, that is, T is a q-strong semilattices of monoids with the inherited operations from S. Trivially, every semilattice with upper bound is a q-strong semilattices of monoids, and hence $E(S)$ is a q-strong semilattices of monoids of S . If the identity 1 of S is improper we clearly have $S = E(S)$ and so S reduces to a semilattice. The converse holds trivially. Thus for any q-strong semilattices of monoids , we have $S \neq E(S)$ if and only if 1 is proper, that is, if and only if S_1 is not the trivial monoid. We observe that, the partial monoid $\overset{\triangle}{P}(X, G)$, for any set X and monoidp G, is a q-strong semilattices of monoids That is for all $B \subset X, \varphi_{X,B} : M(X, G) \to M(B, G)$ is an epimorphism of groups. For, we have $\overset{\triangle}{P}(X, G)$ has the identity element 1_X which is the identity of the maximal monoid $M(X, G)$ and given $f \in M(B, G)$, the mapping $\bar{f} \in M(X, G)$ defined by $\bar{f}(x) = f(x)$ if $x \in B$ and $\bar{f}(x) = 1_G$ if $x \in X - B$ satisfies $\varphi_{X,B}(\bar{f}) = f$. Thus $\varphi_{X,B}$ is an epimorphism. Whence partial mappings (between sets and monoids) may be considered as natural sources of q-strong semilattices of monoids. Clearly $M(X, G)$ is not the trivial monoid if and only if X is non empty and G is not the trivial monoid. Thus $\overset{\triangle}{P}(X, G)$ is a non trivial q-strong semilattices of monoids if and only if $X \neq \varnothing$ and $G \neq 0$. We close this section by one more simple observation.

Lemma 3.1. *If S is a q-strong semilattices of monoids in which no two maximal submonoids are isomorphic, then the kernals of the epimorphisms $\varphi_{1,e}$, $e \in E(S)$, are all different.*

Proof. Let N_e denote the kernal of $\varphi_{1,e}, e \in E(S)$. Thus $N_e = \{y \in S_1 : ye = e\}$. If $e \neq f$ in $E(S)$ and $N_e = N_f$, we have by the first isomorphism theorem of monoids

$$S_e \cong S_1/N_e = S_1/N_f \cong S_f$$

which contradicts the hypothesis. □

4. The Q-Operation

In the previous section we noticed that every strong semilatticess of monoids S with identity contains a trivial q substrong semilattices of monoids, namely, $E(S)$. In this section, we show that nontrivial q substrong semilattices of monoids of S exist whenever S_1 is a non trivial monoid. More precisely, a maximal q substrong semilattices of monoids of S always exists. This inherited to all wide substrong semilattices of monoids T of S, and hence defines an operation $T \to \mathbf{Q}(T)$. In later work, we will show that the $\mathbf{Q}$-operation preserves normality, and commutes with the operation of taking joins. In this section, we show that it commutes with categorical products. Given a wide substrong semilattices of monoids T of S, the existence of $\mathbf{Q}(T)$, the maximal q substrong semilattices of monoids contained in T can be verified by the axiom of choice (e.g. Zorn's lemma), but for later purpose we construct $\mathbf{Q}(T)$ explicitly. It is obtained, simply by taking images of $\varphi_{1,e}$ on T_1 for all $e \in E(S)$. Formally, we have:

Lemma 4.1. *Let S be a strong semilattices of monoids with identity and let T be a wide substrong semilattices of monoids S. There exists a q substrong semilattices of monoids $\mathbf{Q}(T)$ of S which is unique maximal such that $\mathbf{Q}(T) \subset T$. Moreover $\mathbf{Q}(T)$ is non trivial (i.e. doesn't equal $E(S)$) if and only if T_1 is a non trivial monoid.*

Proof. Since T is wide, it is a union of maximal monoids indexed by $E(S)$, that is, $T = [E(S), T_e, \varphi_{e,f}]$ where T_e is a submonoid of S_e and $\varphi_{1,e} : T_1 \to T_e$ is a homomorphism $(x \mapsto xe)$ for every $e \in E(S)$. Define

$$\mathbf{Q}(T) = \underset{e\in E(S)}{\cup} \mathrm{Im}\varphi_{1,e} = \underset{e\in E(S)}{\cup} T_1 e.$$

$\mathbf{Q}(T)$ is a disjoint union of monoids $(\mathbf{Q}(T))_e = T_1 e$ indexed by the semilattice $E(S)$. In particular, $(\mathbf{Q}(T))_1 = \mathrm{Im}\varphi_{1,1} = \mathrm{T}_1$ and the restriction of $\varphi_{1,e}$ on T_1 gives an epimorphism $\varphi_{1,e} : (\mathbf{Q}(T))_1 = T_1 \to (\mathbf{Q}(T))_e$ for every $e \in E(S)$. It follows that $\mathbf{Q}(T)$ is q-substrong semilattices of monoids of S contained in T. For $e > f$, $\varphi_{e,f} : T_1 e \to T_1 f$ is given by $xe \mapsto xf$, $(x \in T_1)$. If K is q-substrong semilattices of monoids of S with $K \subset T$, then $K_1 \subset T_1 = (\mathbf{Q}(T))_1$ and for all $e \in E(S)$, $K_e = K_1 e \subset T_1 e = (\mathbf{Q}(T))_e$. This proves the unique maximality of $\mathbf{Q}(T)$.Finally, $\mathbf{Q}(T) \neq E(S) \Rightarrow (\mathbf{Q}(T))_e \neq \{e\}$ for some $e \in E(S) \Rightarrow T_1 e \neq \{e\} \Rightarrow T_1 \neq \{1\}$, conversely, if $T_1 \neq \{1\}$, then $(\mathbf{Q}(T))_1 = T_1 \neq \{1\}$,
and so $\mathbf{Q}(T) \neq E(S)$. □

Let us now consider strong semilattices of monoids as a part of universal algebra, that is as a variety of algebras (defined by a set of identities). This implies that, as a category, strong semilattices of monoids has all small limits and colimites (e.g. products, coproducts, etc.). This is also true for strong semilatticess of monoids with identities. In the rest of this section we consider categorical products of partial monoidss (with identities) and show that the $\mathbf{Q}$ operation commutes with this product which implies that product of any family of q-strong semilattices of monoids is again a q-strong semilattices of monoids. We start by characterizing products in the category of strong semilattices of monoids.

Lemma 4.2. *Let $\{S_i, i \in I\}$ be a family of strong semilattices of monoids and let $S = \prod_{i\in I} S_i$ be the usual cartesian product. Then S is a strong semilattices of monoids which is a categorical product with the usual projections $\pi_i : S \to S_i$, $(x_i) \mapsto x_i$. If each S_i has an identity 1_{s_i}, then (1_{s_i}) is the identity of S.*

Proof. Define $E(S) = \prod_{i\in I} E(S_i)$. Then $E(S)$ is a semilattice with $(e_i) \leq (f_i)$ if and only if $e_i \leq f_i$ for all $i \in I$. We have

$$S = \prod_{i\in I} S_i = \prod_{i\in I} \bigcup_{e_i\in E(S_i)} (S_i\ )_{e_i} = \bigcup_{(e_i)\in E(S)} \prod_{i\in I} (S_i)_{e_i}$$

Thus S is a disjoint union of monoids $\prod_{i\in I}(S_i)_{e_i}$, $e_i \in E(S_i)$ with identities $(e_i)_{i\in I}, e_i \in E(S_i)$, indexed by the semilattice $E(S)$. For $(e_i) \geq (f_i)$ in $E(S)$, there is a homomorphism

$$\varphi_{(e_i),(f_i)} : \prod_{i\in I}(S_i)_{e_i} \to \prod_{i\in I}(S_i)_{f_i},$$

given by

$$(x_i) \mapsto (\varphi_{e_i,f_i} x_i).$$

Now we can easily verify that S is a categorical product, and that S has identity if each S_i has a one. □

Theorem 4.3. *Let $\{S_i, i \in I\}$ be a family of strong semilattices of monoids, with identities . Then $\mathbf{Q}(\prod_{i\in I} S_i) = \prod_{i\in I} \mathbf{Q}(S_i)$.*

Proof. By Lemma 4.2, $\prod_{i\in I} S_i$ is a strong semilattices of monoids with identity $(1_{s_i})_{i\in I}$ which is a union of maximal submonoids $(\prod_{i\in I} S_i)_{(e_i)}$ indexed by the semilattice $E(S) = \prod_{i\in I} E(S_i)$. By Lemma 4.1, we have

$$\begin{aligned}
\mathbf{Q}(\prod_{i\in I} S_i) &= \bigcup_{(e_i)\in E(S)} (\prod_{i\in I} S_i)_{(1_{s_i})}(e_i) \\
&= \bigcup_{(e_i)\in E(S)} (\prod_{i\in I} (S_i)_{1_{s_i}} e_i) = \prod_{i\in I}(\bigcup_{e_i\in E(S_i)} (S_i)_{1_{s_i}} e_i) \\
&= \bigcup_{(e_i)\in E(S)} (\prod_{i\in I} (S_i)_{1_{s_i}} e_i) = \prod_{i\in I} \mathbf{Q}(S_i).
\end{aligned}$$

□

By the definition of the **Q** operation , one can show that the product of any family of q-strong semilattices of monoids is again a q-strong semilattices of monoids. But if we notice that for any strong semilattices of monoids S with identity , S is a q-strong semilattices of monoids if and only if $\mathbf{Q}(S) = S$, then the following is an easy consequence of Theorem 4.3.

Corollary 4.4. *If $\{S_i, i \in I\}$ is a family of q strong semilattices of monoids, then the product $\prod_{i\in I} S_i$ is a q-strong semilattices of monoids.*

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